Generalisation of bishop polynomial derived from conventional rook polynomial

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1. Abstract

The rook polynomial is a powerful tool in the study of restricted permutations. The bishop polynomial is a subset of the rook polynomial on irregular boards. While the rook polynomial for regular boards is well-established, a generalisation for the bishop polynomial remains unclear (at the point of the study). This study hence presents an original generalisation, the Qu-Sim theorem, for the bishop polynomial for all square boards.

2. Introduction

The rook polynomial is a powerful tool in the theory of restricted permutations. [1] Especially in recent years, the study of rook polynomial branches from the initial focus on the enumeration of restricted permutations to many others, with applications in graph theory, enumeration of matrices, chromatic theory and more. In comparison, the bishop polynomial, a special case of the rook polynomial, has not been as well- established. As such, this paper hopes to present an original generalisation of the bishop polynomial for square boards to help raise the threshold of research on bishop polynomials.

3. Preliminaries

(Note that a glossary of notations is available at the Appendix for easy of reference.)

The rook polynomial is a generating polynomial of the number of arrangements of k non-attacking rooks on an $m \times n$ chessboard. A rook is a chess piece that moves vertically and horizontally on the chessboard. As such, non-attacking rooks must be arranged such that no rook are placed in the same row or column.

Let B denote a generalised board (which may be irregular in shape), whereas $B_{m\times n}$ denote a rectangular (regular) board with m rows and n columns. In this paper, we define the term k-rook placement as the number of arrangements of k non-attacking rooks on a board B whereby order does not matter.

The following are theorems and definitions previously derived by other papers. Proof will not be provided here but can be easily found in the references.

Definition 3.1 *The rook polynomial of a board B is the generating function:* $R_B(x) = \sum_{k=0}^{\infty} r_k(B)x^k$*, where* $r_k(B)$ *denotes the* k*-rook placement on* B.

Lemma 3.1 $r_k(B_{m\times n}) = \binom{m}{k}\binom{n}{k}k!$ *where* $k \geq m, n$.

Theorem 3.2 *The rook polynomial of* $B_{m\times n}$ *is defined as:* $R_{m,n}(x) = \sum_{k=0}^{\min(m,n)} \binom{m}{k}\binom{n}{k}\ k!\ x^k$ *while a 0-rook placement on any board is always 1.*

Definition 3.2 *Two boards,* A *and* B*, are said to be rook equivalent if* $R_A(x) = R_B(x)$*. A sufficient (while not always necessary) condition of rook equivalence is that* B *can be obtained from* A *by permutation of rows and columns, meaning interchanging rows and columns will not alter the* k*-rook placements of a board.[1]*

Theorem 3.3 *Component Theorem Let* B *be a board consisting of* k *disjoint subboards,* $B_1, B_2, B_3 \ldots B_k$*. Then* $R_B(x) =$

$\Pi_{i=1}^{k} R_{B_k}(x)$, *where disjoint sub-boards are defined as boards made up of cells from the original board B not sharing common rows/columns.[2]*

Definition 3.3 *Let $B_{m\times n}$ be an $m\times n$ board, and S an $s\times t$ block of $B_{m\times n}$. For $0 \leq j \leq \min(s,t)$, let $B_{S,j}$ denote the board obtained by deleting*

- *j of the s rows covered by S*
- *j of the t columns covered by S*
- *all the cells of S*

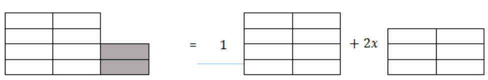
$B_{S,j}$ is known as the j^{th} inclusion board of $B_{m\times n}$ relative to S. Since rows and columns involved are identical except on S, which are deleted, $B_{S,j}$ is well-defined. [1]

Theorem 3.4 *Block Decomposition Theorem [1]*

$$R(B) = \sum_{j=0}^{\min(s,t)} r_k(S)x^j R(B_{S,j})$$

(Efficiency of the theorem is greatly affected by the choice of cell/block to decompose.)

Example: In Figure 1, let S be the 2×1 shaded block. Then by Theorem 2.4, the rook polynomial of un-decomposed board B is $r_0(S)R(B_{s,0}) + r_1(S)R(B_{s,1})$ as shown.



Example of Application of Block Decomposition

4. Special case of Rook Polynomial: $2 \times n$ boards

This special case of the Rook Polynomial will be used intensively in our proof of our main theorems presented, accompanied with Theorem 2.4. As such, it is instructive to go through the general formula of the rook polynomial for $n \times 2$ boards.

Corollary 4.0.1 *The rook polynomial for $n \times 2$ boards is as follows: $R_{n\times 2}(x) = 1+(n\times 2)x+(n)(n-1)x^2$ where $R_{n\times 2}(x)$ denotes the rook polynomial for $n \times 2$ boards.*

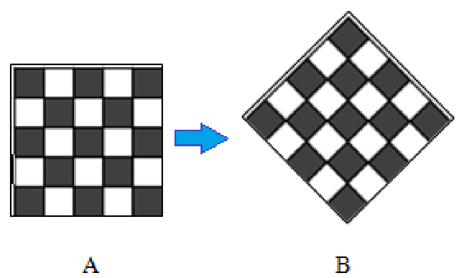
Proof 4.0.1 *: Since there is always 1 way to arrange 0 rooks on any board. Consider all possible $n \times 2$ boards, using Theorem 2.2 there is $\binom{n}{1}\binom{2}{1} = 2n$ ways to arrange 1 non-attacking rook; and $\binom{n}{2}\binom{2}{2}2! = (n)(n-1)$ ways to arrange 2 non-attacking rooks.*

5. Introduction to the Bishop Polynomial

A bishop moves only diagonally without restriction in the distance of each move.

Definition 5.1 *Here, the bishop polynomial is the generating function of the number of arrangements of k non-attacking bishops (k-bishop placement) on a $m \times n$ board: $B_{m\times n}(x) = \sum_{k=0} b_k x^k$ where b_k denotes the k^{th} coefficient of the bishop polynomial.*

While both bishops and rooks move in straight lines, they differ in the direction of movement. However, the movement of the two can be related through a 45^o rotation of the board. Tracing out the path of a bishop after a 45^o rotation, gives the path of a rook piece. In Figure 2, the bishop polynomial of board A is the rook polynomial of board B.



Tilting board 45^o converts bishop move (in A) to a rook one (in B)

Given that a rook moves only vertically or horizontally, it can only occupy squares of the same colour (in board B above). Thus the white board (consisting of white cells) and black board (consisting of black cells) are disjoint sub-boards of the overall board B. In particular, for even boards $B_{2n\times 2n}$, note that $R(B_{\text{white}}) = R(B_{\text{black}})$. By Theorem 2.3, $R(B) = R(B_{\text{white}}) \times R(B_{\text{black}})$ for all square board B.

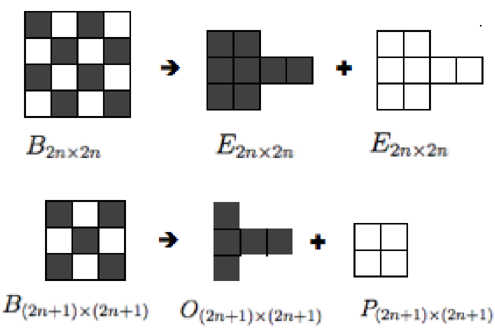
6. The Qu-Sim Theorem

6.1. Definitions

Definition 6.1 *Square even boards, $B_{2n\times 2n}$ can be decomposed into two rook equivalent sub-boards, each denoted as E board or $E_{2n\times 2n}$. Bishop polynomial of $E_{2n\times 2n}$ is expressed as $E_{2n\times 2n}(x)$. Based on Theorem 2.3, $B_{2n\times 2n}(x) = (E_{2n\times 2n}(x))^2$*

Definition 6.2 *Square odd boards, $B_{(2n+1)\times(2n+1)}$ can be broken down into two sub-boards: one with even number of cells – denoted as P board or $P_{(2n+1)\times(2n+1)}$ - and the other with an odd number of cells – denoted as O board or $O_{(2n+1)\times(2n+1)}$.*

Bishop polynomial of the P and O boards are expressed as $P_{(2n+1)\times(2n+1)}(x)$ and $O_{(2n+1)\times(2n+1)}(x)$. By Theorem 2.3, $B_{(2n+1)\times(2n+1)}(x) = P_{(2n+1)\times(2n+1)}(x) \times O_{(2n+1)\times(2n+1)}(x)$



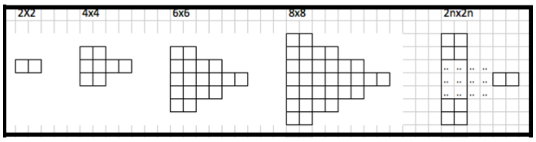
Decomposition of $B_{2n\times 2n}$ and $B_{(2n+1)\times(2n+1)}$

6.2. Qu-Sim Theorem for Even Boards

Theorem 6.1 $B_{2n\times 2n}(x) = (\sum_{k=0}^{2n-3} x^k\, b_k(E_{(2n-2)\times(2n-2)}(x)) \times R_{2\times(2n-k-1)}(x))^2$

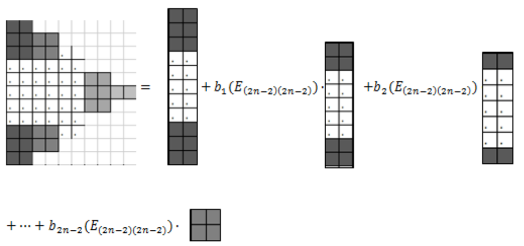
where $n \geq 2$ and $E_{2\times 2}(x) = 1 + 2x$. Note that $b_k(B(x))$ refers to the k^{th} bishop coefficient of bishop polynomial of sub-board $E(x)$ and $R_{m\times n}(x)$ refers to the rook polynomial of regular board $B_{m\times n}$.

Proof 6.1.1 *Transforming the first few even boards (Figure 4) gives us a general idea of the pattern: The sub-boards $E_{(2n+2)\times(2n+2)}$ always adds a $2 \times (2n+1)$ block to the former E sub-board $E_{2n\times 2n}$. Hence, as the consecutive boards are closely related, a recursive function can be used to calculate the bishop polynomial for each board.*



First few E boards (subboard of the even board $B_{2n\times 2n}$)

Using Theorem 2.4, we see that for $E_{2n\times 2n}$ the following relations hold if we choose to decompose the previous $E_{(2n-2)\times(2n-2)}$ board: (Refer to Figure 5 for diagrammatic representation of relations.)



Block Decomposition of the $E_{2n\times 2n}$ board) (which is the board on the LHS of equation)

Representing Figure 5 mathematically,

$B_{2n\times 2n}(x) = b_0(E_{2n-2)\times(2n-2)}(x)) \times R_{(2n-1)\times 2}(x) + b_1(E_{(2n-2)\times(2n-2)}(x)) \times R_{(2n-2)\times 2}(x) + b_2(E_{(2n-2)\times(2n-2)}(x)) \times$

$R_{(2n-4)\times 2}(x) \quad x^2 \quad + \quad \dots \quad + \quad b_{2n-2}(E_{(2n-2)\times(2n-2)}(x))\times R_{(3\times 2)}(x)\, x^{2n-2} + b_{2n-3}(E_{(2n-2)\times(2n-2)}(x))\times R_{2\times 2}(x)\, x^{2n-3}$

which is equivalent to the Qu-Sim Theorem presented at the start of this section.

6.2.1. Defining the Bishop Coefficient of E Boards

Definition 6.3 *The k^{th} bishop coefficient of the E sub-board is defined as follows:*

$b_k(E_{2n\times 2n}(x)) = b_k(E_{(2n-2)\times(2n-2)}(x)) + 2(2n-k)\cdot b_{k-1}(E_{(2n-2)\times(2n-2)}(x)) + (2n-k+1)(2n-k)\cdot b_{k-2}(E_{(2n-2)\times(2n-2)}(x))$

Where $n \geq 2,\ 2 \leq k \leq 2n-1,\ n,k \in Z$ and $b_1(E_{2\times 2}) = 2$. Note for all $k > 2n-1,\ b_k(E_{2n\times 2n}(x)) = 0$; and for all $k > 2n-3,\ b_k(E_{(2n-2)\times(2n-2)}(x)) = 0$.

The explicit forms of b_0, b_1, b_2 for all square boards (even and odd) are also found and presented in Section 6 of the paper.

Proof 6.1.2 *First let's decompose the board $E_{2n\times 2n}$ into its sub-board $E_{(2n-2)\times(2n-2)}$ and the additional part which is a $2\times(2n-1)$ rectangular board. We can place $0, 1$ or 2 bishops on the $2\times(2n-1)$ board, with a corresponding $k, k-1$ or $k-2$ bishops on $E_{(2n-2)\times(2n-2)}$, yielding in total three cases. Given that in order for the bishops on the additional board to be non-attacking in relation to the previous ones, there are $1, 2n-k$ and $(2n-k+1)(2n-k)$ ways respectively. Then, it follows from Theorem 2.4 (Decomposition Theorem) that:*

$b_k(E_{2n\times 2n}(x)) = b_k(E_{(2n-2)\times(2n-2)}(x)) + 2(2n-k)\cdot b_{k-1}(E_{(2n-2)\times(2n-2)}(x)) + (2n-k+1)(2n-k)\cdot b_{k-2}(E_{(2n-2)\times(2n-2)}(x))$ *as above.*

6.3. Qu-Sim Theorem for Odd Boards

Theorem 6.2 $B_{(2n+1)\times(2n+1)}(x) = \left[\sum_{k=0}^{2n-2} x^k \quad b_k(P_{(2n-1)\times(2n-1)}(x)) \times R_{2\times(2n-k)}(x)\right]\left[\sum_{k=0}^{2n-1} x^k\, b_k(O_{2n\times 2n}(x)) \times R_{1\times(2n-k+1)}(x)\right]$

where $n \geq 2$. Note that $b_k(B(x))$ refers to the k^{th} bishop coefficient of some bishop polynomial of sub-boards $P(x)$ or $O(x)$.

6.3.1. Defining the Bishop Coefficient of P Boards

Definition 6.4 *The k^{th} bishop coefficient of the P sub-board is defined as follows:*

$b_k(P_{(2n-1)\times(2n-1)}(x)) = b_k(P_{(2n-3)\times(2n-3)}(x)) + 2(2n-k-1)\cdot b_{k-1}(P_{(2n-3)\times(2n-3)}(x)) + (2n-k-1)(2n-k)\cdot b_{k-2}(E_{(2n-3)\times(2n-3)}(x))$

where $n \geq 3, 2 \leq k \leq 2n-3, k \in Z$ and $b_1(P_{1\times 1}) = 1, b_1(P_{3\times 3}) = 4, b_2(P_{3\times 3}) = 2$. Note that $b_0 = 1, b_1(P_{(2n-1)\times(2n-1)}(x)) = 2\times n\times(n-1)\ for\ n \geq 2$, and for all $k > 2n-2,\ b_k(P_{(2n-1)\times(2n-1)}(x)) = 0$.

The proof of Theorem 5.2 and Definition 5.4 are included in the Appendix. Note a table of bishop polynomials up to 8×8 boards is available in the Appendix.

7. Explicit Forms - Special cases of $b_k(B_{n\times n})$

By definition, $b_0 = 1$ for all boards. It is also clear that $b_1 = n^2$ for all square boards $B_{n\times n}$) since there are n^2 cells in total and with only 1 bishop, there are no restrictions on the placement of bishop.

Lemma 7.1 *Closed Form of b_2:* $b_2(B_{n\times n}) = \frac{1}{6}n(n-1)(3n^2-n+2)$.

The proof of the Lemma is also included in the Appendix of the report.

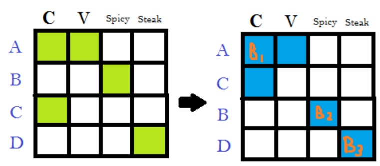
8. Application and Conclusion

The rook polynomial is pertinent in various fields of combinatorics, including counting problems with restrictions and graph theory (matching polynomials).

Matching polynomial, in particular, has known applications in Chemistry. For instance, the Hosoya Z topological index correlates to the boiling points of alkanes and is calculated to be the sum of coefficients of the matching polynomial of the graph of the alkane molecule. The rook polynomial is particularly useful when dealing with bipartite graphs which can be modelled like boards in the rook problem.

The rook polynomial, when paired with the Inclusion-Exclusion Theorem, can solve real-life counting problems such as timetable pairing and allocation of resources.

For example: A restaurant provides these set meals: Chicken(C), Vegetable(V), Spicy and Steak. Adam(A) doesn't eat chicken and vegetable; Ben(B) doesn't eat spicy food; Carrie(C) doesn't eat chicken; and David(D) doesn't eat beef. In how many ways can Mum order a different set meal for each of the children such that all end up satisfied?



On the left is rook-equivalent board (to that of the right) after permutation of rows and columns to obtain 3 disjoint sub-boards.

In Figure 6, the shaded squares refer to the pairing of children to set meal that will not work. Here, the rook polynomial comes in handy since the number of ways to satisfy the children is just the complement of the rook polynomial of the shaded set of cells.

Additionally, for incomplete square $n \times n$ boards (i.e. rooks cannot be placed on subsets of the board), the k^{th}- rook placement is equivalent to computing the permanent of a 0-1 matrix. Our study of bishop polynomial (an irregular rook problem) may shed more light in this field.

The bishop polynomial also has practical and theoretical application:

Given a room with diagonal corridors, how many ways can k robots be arranged such that their arrangement has no overlaps (no 2 robots are placed in same diagonal)? Problems like these can be solved using the bishop polynomial, which can help optimise real-life guarding system.

In addition, the bishop polynomial can be related to theoretical fields such as chromatic polynomial, determinants as well as discordant permutations (derangement). These applications are discussed further in the paper by Wahid (1999). [3]

The bishop polynomial, an irregularity of rook polynomial, is not easy to analyse combinatorially. Given the better-established rook polynomial, the difference in movement of pieces can be resolved by rotation of the board. The problem is then reduced to solving for the rook polynomial of an irregular board. In our report, patterns of shapes of sub-boards and of successive boards are used for generating the Qu-Sim Theorem, an original generalisation of the bishop polynomial.

We endeavour to expand our project in the following ways: First, to define the the bishop polynomial without recurrence, instead with an explicit. In other words, we hope to find a closed-form solution for the bishop polynomial.

Second, we hope to expand the generalisation of the bishop polynomial using the method previously discussed on other board such as rectangular or triangular boards.

Lastly, we can investigate the bishop polynomial of 3-dimensional boards as the rook polynomial of 3D boards are already well-established.

9. Appendix

9.1. Proof of Qu-Sim Theorem for Odd Board

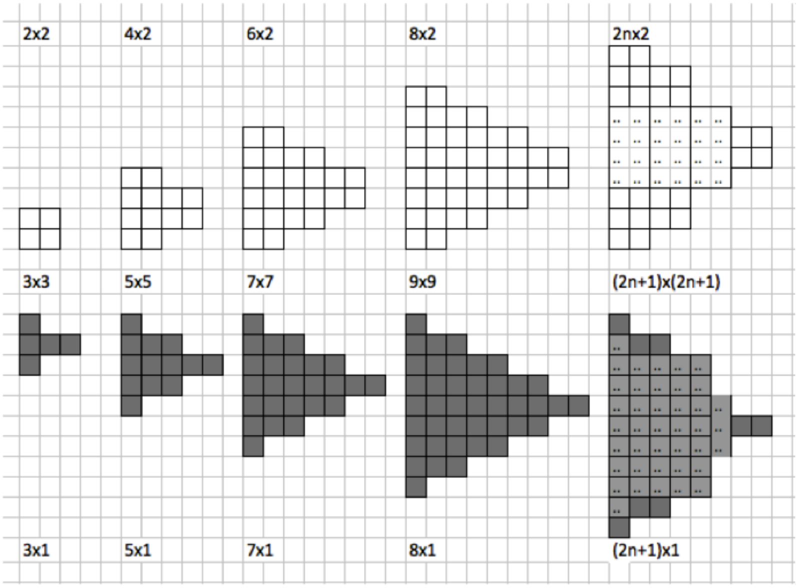


Figure 1. First few P boards(upper) and O boards(lower)

Proof 9.0.1 *Transforming the first few odd boards gives us a general idea of the pattern: For P boards, the sub-boards $P_{(2n+1)\times(2n+1)}$ always adds a $2 \times 2n$ block to the former P sub-board $P_{(2n-1)\times(2n-1)}$. For O boards, the sub-boards $O_{(2n+1)\times(2n+1)}$ always adds a $1 \times n$ block to sub-board $E_{2n\times 2n}$. A recursive function function can thus be formed for both P and O boards.*

Using Theorem 2.4, the following holds if we decompose $P_{(2n+1)\times(2n+1)}$ into its previous boards.

Representing Figure 7 mathematically,

$P_{(2n+1)\times(2n+1)} = 1 \times R_{2\times 2n}(x) + b_1(P_{(2n-1)\times(2n-1)}(x)) \times x \times R_{2\times(2n-1)}(x) + b_2(P_{(2n-1)\times(2n-1)}(x)) \times x^2 \times R_{2\times(2n-1)}(x) + ... + b_{2n-3}(P_{(2n-1)\times(2n-1)}(x)) \times x^(2n-3) \times R_{2\times 3}(x) + b_{2n-2}(P_{(2n-1)\times(2n-1)}(x)) \times x^(2n-2) \times R_{2\times 2}(x)$

which is equivalent to $P_{(2n+1)\times(2n+1)}(x) = \left[\sum_{k=0}^{2n-2} x^k \ b_k(P_{(2n-1)\times(2n-1)}(x)) \times\right.$

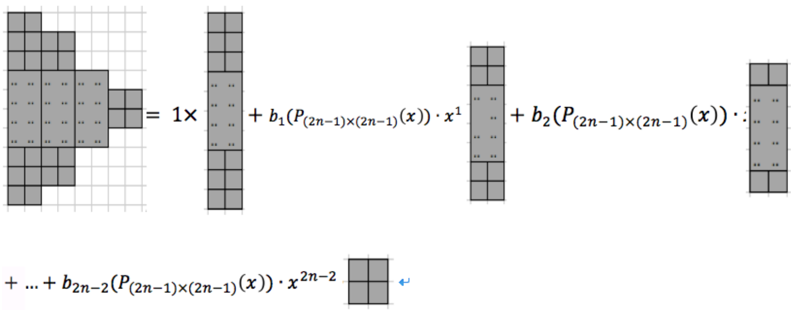


Figure 2. Block Decomposition of $P_{(2n+1)\times(2n+1)}$

$R_{2\times(2n-k)}(x)\big]$ $(n, k \in Z, n \geq 2)$

Similarly, $O_{(2n+1)\times(2n+1)}(x) = \big[\sum_{k=0}^{2n-1} x^k\, b_k(E_{(2n)\times(2n)}(x)) \times R_{2\times(2n-k+1)}(x)\big]$ $(n, k \in Z, n \geq 2)$

9.2. Definition of P board coefficient

Substituting $(2n-1)$ for all $2n$ in section $5.2.1$, we get:

$b_k(P_{(2n-1)\times(2n-1)}(x)) = b_k(P_{(2n-3)\times(2n-3)}(x)) + 2 \times (2n-k-1) \times b_{k-1}(P_{(2n-3)\times(2n-3)}(x)) + (2n-k) \times (2n-k-1) \times b_{k-2}(P_{(2n-3)\times(2n-3)}(x))$

where $n \geq 3, 2 \leq k \leq 2n-3$ and $b_1(P_{1\times1}) = 1, b_1(P_{1\times1}) = 4, b_2(P_{3\times3}) = 2$. Note for all $k > 2n-2, b_k(P_{(2n-1)\times(2n-1)}(x)) = 0$

9.2.1. Special Cases of $k = 0, 1$

$b_0 = 1, b_1(P_{(2n-1)\times(2n-1)}(x)) = 2 \times n \times (n-1)$ Holds for all $n \geq 2$.

9.3. Proof of Closed Form for b_2

Proof 9.0.2 *Referring to Figure 7, construct the previous board $B_{(n-1)\times(n-1)}$ as a subboard within the board $B_{n\times n}$. Then we see that by doing so, we can construct a recurrence relations between b_2 of the 2 consecutive boards. To place 2 non-attacking bishops on $B_{n\times n}$, there are 3 cases:*

1. *2 bishops in the subboard constructed, $B_{(n-1)\times(n-1)}$*
2. *1 bishop in $B_{(n-1)\times(n-1)}$ and the other 1 outside $B_{(n-1)\times(n-1)}$ but in $B_{n\times n}$*
3. *2 bishops placed on cells outside $B_{(n-1)\times(n-1)}$ but in $B_{n\times n}$*

Case 1: *Number of ways satisfying Case 1 is simply $b_2(B_{(n-1)\times(n-1)}(x))$*
Case 2: *We shall enumerate this by finding the complement:*
Total number of ways to place 1 bishop outside subboard and 1 inside subboard without restriction = $\binom{(n-1)^2}{1}\binom{2n-1}{1} = 2n^3 - 5n^2 + 4n - 1$.
Total number of ways to place 2-attacking bishops, 1 in subboard and 1 outside = $\frac{3(n-1)(n-2)}{2} + \frac{n(n-1)}{2} = 2n^2 - 5n + 3$.
Thus, number of ways satisfying Case 2 = $2n^3 - 5n^2 + 4n - 1 - (2n^2 - 5n + 3) = 2n^3 - 7n^2 + 9n - 4$.
Case 3: *Number of ways satisfying Case 3 = $\binom{2n-2}{2} \cdot 1 + (\frac{\binom{2n-2}{1}\binom{2n-4}{1}}{2}) = 2n - 2 + 2n^2 - 6n + 4 = 2n^2 - 4n + 2$.*
Therefore, the recurrence relations we have derived is:

$$\begin{aligned} b_2(B_{n\times n}(x)) &= b_2(B_{(n-1)\times(n-1)}(x)) + 2n^3 - 7n^2 + 9n - 4 + 2n^2 - 4n + 2 \\ &= b_2(B_{(n-1)\times(n-1)}(x)) + 2n^3 - 5n^2 + 5n - 2 \end{aligned}$$

Now, for convenience's sake, let $a_n = b_2(B_{n\times n}(x))$, then $a_n = a_{n-1} + 2n^3 - 5n^2 + 5n - 2$. Removing all constants and terms except for a_n, we get:

$$a_n - 5a_{n-1} + 10a_{n-2} - 10a_{n-3} + 5a_{n-4} - a_{n-5} = 0.$$

Using the method of characteristics equation we obtain: $x^5 - 5x^4 + 10x^3 - 10x + 5x - 1 = (x-1)^5 = 0$. Clearly the 5 roots of this equation are $1, 1, 1, 1, 1$, then forming generating functions, we get:

$$a_n = (c_0 + c_1 n + c_2 n^2 + c_3 n^3 + c_4 n^4) \cdot 1^n$$

. *Solving a system of equation, we get that* $(c_0, c_1, c_2, c_3, c_4) = (0, -\frac{1}{3}, \frac{1}{2}, -\frac{2}{3}, \frac{1}{2})$. *Hence,*

$$\begin{aligned} a_n &= -\tfrac{1}{3}n + \tfrac{1}{2}n^2 - \tfrac{2}{3}n^3 + \tfrac{1}{2}n^4 \\ &= \tfrac{1}{6}n(n-1)(3n^2 - n + 2) \end{aligned}$$

as above.

9.4. Verification of Qu-Sim

The below is a computer programmed code in C++ that counts the number of k bishop placements even an $n \times n$ board.[4] Figure 9 shows the bishop polynomial for square boards up to dimensions 8×8 derived from the Qu-Sim which has been verified with the values generated from the computer program (code attached below for reference).

```
#include <iostream>
using namespace std;

const int N = 70;

/*
* c1 and c2 represent the number of cells in rows of the 2 sub-boards
* (white and black) of the board
* (By Component Theorem,
* we can simply multiply the bishop polynomial of both sub-boards
* to obtain the overall bishop polynomial.)
*
* dp1[i][j] and dp2[i][j] are number of ways
* to place j bishops onto the first i
* lines of the respective sub-boards (c1 and c2)
*/

int c1[N], c2[N], dp1[N][N], dp2[N][N];

/*
* Initializes the number of cells in a row after rotation (45 deg)
* For example: If n is 3, all the items in c1[] and c2[] are:
* 1, 2, 3, 2, 1 (though they may not be sorted)
* Essentially the function decomposes the board into 2.
*/

void init(int n, int *c1, int *c2)

{
    for (int i=1; i<=n; i++)
        for (int j=1; j<=n; j++)
        {
            if((i+j)%2)
                c2[(i+j)/2]++;
            else
```

```
                c1[(i+j)/2]++;
        }
}

/*
* Calculates the number of ways j bishops can be placed in:
* the first i lines of the c1 or c2.
*
* The logic behind this function is based on a recursion:
* Let dp[i][j] be the no. of ways to place all j bishops on the first
* i lines.
*
* Then we can represent it recursively as such:
* dp[i][j] = dp[i-1][j] + dp[i-1][j-1]*(c[i]-j+1);
*
* In words, dp[i][j] = no. of ways to place j bishops on the first
* i - 1 lines + no. of ways to place j - 1 bishops on the first i - 1
* lines multiplied by the number of ways to place the last bishop
* on that row in particular (given by (c[i]-j+1))
*
* From the above logic (using c[i] in the definition of dp [i][j])
* we see that c1 and c2 must be sorted.
*/


void bishops(int n, int dp[N][N], int c[N])
{
    int i,j;
    for (i=0; i<=n; i++)

        // since 0-bishop placement is always equals to 1
        dp[i][0]=1;

    for (i=1; i<=n; i++)
        for (j=0; j<=c[i]; j++)
            dp[i][j] = dp[i-1][j] + dp[i-1][j-1]*(c[i]-j+1);
}

int main()
{
    int n, k, ans, i;
    while (cin >> n >> k) {
        if (n==0 && k==0) break;

        // initiate c1[] and c2[]
        init(n, c1, c2);
```

```
        // Since items in c1 and c2 are not sorted, we proceed to sort
        sort(c1+1, c1+n+1);
        sort(c2+1, c2+n);


        // call bishops function for both c1 and c2
        bishops(n, dp1, c1);
        bishops(n, dp2, c2);

        ans=0;

        /*
        * When i bishops are placed on one board,
        * k - i bishops (remaining ones) must be on the other.
        */

        for (i=0; i<=k; i++)
            ans += dp1[n][i]*dp2[n-1][k-i];

        cout << ans << endl;
    }
    return 0;
}
```

n represents the size of the board ($n * n$ board); k represents the number of bishops. The outputs of this code corroborate with the bishop polynomial (listed in the table below) derived from Qu-Sim Theorem

$2n+1$	$P_{(2n+1)\times(2n+1)}(x)$	$O_{(2n+1)\times(2n+1)}(x)$	$B_{(2n+1)\times(2n+1)}(x)$
1	0	$x+1$	$x+1$
3	$2x^2+4x+1$	$4x^2+5x+1$	$8x^4+26x^3+26x^2+9x+1$
5	$4x^4+32x^3+38x^2+12x+1$	$8x^4+46x^3+46x^2+13x+1$	$32x^8+440x^7+1960x^6+3368x^5+2728x^4+1124x^3+240x^2+25x+1$
7	$8x^6+208x^5+652x^4+576x^3+188x^2+24x+1$	$16x^6+308x^5+836x^4+674x^3+206x^2+25x+1$	$128x^{12}+5792x^{11}+81184x^{10}+389312x^9+867328x^8+1022320x^7+692320x^6+282248x^5+70792x^4+10894x^3+994x^2+49x+1$

$2n$	$E_{2n\times 2n}(x)$	$B_{2n\times 2n}(x)$
2	$2x+1$	$4x^2+4x+1$
4	$4x^3+14x^2+8x+1$	$16x^6+112x^5+260x^4+232x^3+92x^2+16x+1$
6	$8x^5+100x^4+184x^3+98x^2+18x+1$	$64x^{10}+1600x^9+12944x^8+38368x^7+53744x^6+39680x^5+16428x^4+3896x^3+520x^2+36x+1$
8	$16x^7+632x^6+2816x^5+3532x^4+1704x^3+356x^2+32x+1$	$256x^{14}+20224x^{13}+489536x^{12}+3672448x^{11}+12448832x^{10}+22057472x^9+22522960x^8+14082528x^7+5599888x^6+1444928x^5+242856x^4+26192x^3+1736x^2+64x+1$

Figure 3. Even Boards: $2n \times 2n$

9.5. Glossary of Notations

Here, a list of the notations mentioned earlier (and are used repeatedly) is compiled for the ease of reference. They are further categorised for convenience.

9.5.1. General Terms

1. B refers to a board (may or may not be regular); $B_{m\times n}$ denotes a board with m rows and n columns.

9.5.2. Rook Polynomial

2. $R_{m\times n}(x)$ denotes the rook polynomial of $B_{m\times n}$; $R(B)$ refers to the rook polynomial of board B [Note: Both notations are used interchangeably.]
3. r_k denotes the k^{th} rook coefficient of $R_{m\times n}(x)$

9.5.3. Bishop Polynomial

4. $B_{m\times n}(x)$ denotes the bishop polynomial of regular $B_{m\times n}$ (in this paper)
5. b_k refers to the k^{th} bishop coefficient of $B_{m\times n}(x)$
6. For regular even boards $B_{2n\times 2n}$, the 2 identical –rook equivalent - even sub-boards are denoted as $E_{2n\times 2n}$; whereas the bishop polynomial of the sub-board is denoted as $E_{2n\times 2n}(x)$.
7. For regular odd boards $B_{(2n+1)\times(2n+1)}$, the even and odd sub-boards are denoted as $P_{(2n+1)\times(2n+1)}$ and $O_{(2n+1)\times(2n+1)}$ respectively; whereas the rook polynomial of the even and odd sub-boards are denoted as $P_{(2n+1)\times(2n+1)}(x)$ and $O_{(2n+1)\times(2n+1)}(x)$ respectively.

References

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